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ABSTRACT. In this article we introduce and develop a theory of tensor products of JW-algebras. Since JW-algebras are so close to W*-algebras, one can expect that the W*-algebra tensor product theory will be actively involved. It is shown that if M and N are JW-algebras with centres Z1 and Z2 respectively, then Z1 ⊗ Z2 is not the centre of the JW-tensor product JW(M ⊗ N) (see below for notation) of M and N, in general. Also, the type decomposition of JW(M ⊗ N) has been determined in terms of the type decomposition of the JW-algebras M and N which, essentially, rely on the relationship between the types of the JW-algebra and the types of its universal enveloping Von Neumann algebra.

1. Introduction. Throughout this paper we rely on the theory of tensor products of Von Neumann algebras. Our standard references will be [11, 13, 14, 15, 20]. If A and B are C*-algebras, then the minimal C*-tensor product of A and B is denoted by A ⊗min B. Given Von Neumann algebras M and N acting on Hilbert spaces H and K, respectively, the Von Neumann tensor product of M and N is the Von Neumann algebra generated by the algebraic tensor product M ⊗ N of M and N in $\mathcal{B}(H \otimes K)$. It is known that if M and N are W*-algebras, then there is a unique central projection Z in $(M \otimes \min N)^{**}$ such that $M \otimes \min N$ is identified with a weak-* dense C*-subalgebra of $(M \otimes \min N)^{**}Z$. The W*-algebra $(M \otimes \min N)^{**}Z$ is called the W*-tensor product of M and N, and is denoted by $M \otimes N$ [15, p. 66], [20, p. 221]. If $\pi_1: M \rightarrow \mathcal{B}(H)$ and $\pi_2: N \rightarrow \mathcal{B}(K)$ are faithful representations of W*-algebras, then the W*-tensor product of M and N is isomorphic to the Von Neumann tensor product of $\pi_1(M)$ and $\pi_2(N)$. Based on this fact, the Von Neumann tensor product of two Von Neumann algebras M and N will be, also, denoted by $M \otimes N$ without specifying the Hilbert spaces on which M and N act.

If M is a JW-algebra, let $W^*(M)$ be the universal enveloping Von Neumann algebra of M, and let $\phi_M$ be the canonical involutory *-antiautomorphism of $W^*(M)$. Usually we will regard M to be a generating Jordan subalgebra of $W^*(M)$ so that $\phi_M$ fixes each point of M. The real Von Neumann algebra $RW^*(M) = \{x \in W^*(M) : \phi_M(x) = x^*\}$ satisfies $RW^*(M)^{**} \cong W^*(M)$ and $W^*(M) = RW^*(M) \oplus i RW^*(M)$. If M is a JC-algebra, let $C^*(M)$ be the universal enveloping C*-algebra of M. It is known that $C^*(M)^{**} \cong W^*(M^{**})$. The reader is referred to [9, Chapter 7] for the properties of $C^*(M)$ and $W^*(M)$.

Let M be a JW-algebra. M is said to be finite if every family of orthogonal equivalent projections is finite. A projection e in M is said to be finite if eMe is a finite JW-algebra and to be abelian if eMe is abelian. Further, M is said to be of
(a) **Type I** if \( c_M(e) = 1 \) for some abelian projection \( e \) of \( M \), where \( c_M(e) \) is the central support of \( e \) in \( M \).

(b) **Type I_{fin}** if \( M \) is of Type I and finite.

(c) **Type I_{co}** if \( M \) is of Type I with no non-zero central finite projections.

(d) **Type II_{1}** if \( M \) is finite without non-zero abelian projections.

(e) **Type II_{co}** if \( c_M(e) = 1 \) for some finite projection \( e \) of \( M \), and \( M \) has no Type I or II_{1} summand.

(f) **Type III** if \( M \) has no non-zero finite projections.

Every JW-algebra decomposes into a direct sum of some or all of the five types (b)--(f) [21, Theorem 13].

Given \( n < \infty \) and orthogonal equivalent abelian projections \( e_1, \ldots, e_n \) in \( M \) such that \( \sum_{i=1}^{n} e_i = 1 \), \( M \) is said to be of **Type I_{n}**; and \( M \) is of Type I_{fin} if and only if \( M = \sum_{n \in S} M_n \), where \( S \subseteq \mathbb{N} \) (possibly infinite) and each \( M_n \) is a Type I_{n} JW-algebra.

In addition, the **Type I_{n}** JW-algebras \((n < \infty)\) decompose into distinct types. In order to describe these we will say that:

(i) \( M \) is of Type I_{n,F}, where \( n < \infty \) and \( F = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \) if \( M \simeq C(X,M_n(F)_{\text{sa}}) \), for some compact hyperstonean space \( X \).

(ii) \( M \) is of Type I_{2,k} if every factor representation of \( M \) is onto the spin factor \( V_k \). If \( k < \infty \), this means that \( M = C(X,V_k) \) for some compact hyperstonean space \( X \); and when \( k \) is an infinite cardinal, it is equivalent to the existence of a weak-* dense JC-subalgebra in \( M \) of the form \( C(X,V_k) \), for some compact Hausdorff space \( X \) (Stacey [17]).

If \( M \) is of Type I_{n}, \( 3 \leq n < \infty \), then \( M \) is a direct sum of the three types described in (i). If \( M \) is of Type I_{2} then there exists a subset \( S \subseteq \mathbb{N} \) and a set \( S_{\infty} \) of infinite cardinals such that

\[
M = \sum_{k \in S} \oplus M_k \oplus \sum_{k \in S_{\infty}} \oplus M_k
\]

where each \( M_k \) is of Type I_{2,k}. We will call \( \sum_{k \in S_{\infty}} \oplus M_k \) the **Type I_{2,\infty}** part of \( M \) (see the proof of Theorem 2.7 of [4], references [9, 16, 17, 21]).

Let \( A \) and \( B \) be JC-algebras. We may suppose that \( A \) and \( B \) are canonically embedded in their respective universal enveloping C*-algebras \( C^*(A) \) and \( C^*(B) \). The completion \( JC(A \otimes_{\min} B) \) of the real Jordan algebra \( J(A \otimes B) \) generated by \( A \otimes B \) in \( C^*(A) \otimes_{\min} C^*(B) \) is called the **JC-tensor product of \( A \) and \( B \) with respect to the minimal C*-norm on \( C^*(A) \otimes C^*(B) \).** For a detailed account of the theory of tensor products of JC-algebras, the reader is referred to [5].

2. **Tensor products of JW-algebras.** Each JW-algebra sits inside a Von Neumann algebra so it is natural to seek means of exploiting the theory of tensor products of Von Neumann algebras in order to produce a theory of tensor products of JW-algebras. In this section, we introduce the definition of the tensor product of two JW-algebras. Theorem 2.9 is the main result of this section, and it provides a useful tool in the succeeding sections.
DEFINITION 2.1. Let $M \subseteq \mathcal{B}(H)_{s.a}$ and $N \subseteq \mathcal{B}(K)_{s.a}$ be JW-algebras, where $H$ and $K$ are complex Hilbert spaces. We denote by $J_{H,K}(M \otimes N)$ the Jordan subalgebra of $\mathcal{B}(H \otimes K)_{s.a}$ generated by the (real) algebraic tensor product $M \otimes N$ of $M$ and $N$. The corresponding weak-operator closure will be denoted by $JW_{H,K}(M \otimes N)$. Then $JW_{H,K}(M \otimes N)$ is a JW-algebra, and is called the JW$_{H,K}$-tensor product of $M$ and $N$. If $[M]^-, [N]^-$ denote the Von Neumann subalgebras of $\mathcal{B}(H)$, $\mathcal{B}(K)$ generated by $M$, $N$ respectively, then by definition, the Von Neumann algebra $[JW_{H,K}(M \otimes N)]^-$ generated by $JW_{H,K}(M \otimes N)$ in $\mathcal{B}(H \otimes K)$ is $[M]^- \otimes [N]^-$.

DEFINITION 2.2. Given JW-algebras $M$ and $N$, regarded as canonically embedded in $W^*(M)$ and $W^*(N)$, respectively, we define the JW$_{-}$-tensor product of $M$ and $N$ to be the JW-algebra generated by $M \otimes N$ in $W^*(M) \otimes W^*(N)$. We denote it by $JW(M \otimes N)$.

Let $M$ and $N$ be JW-algebras. By [4, Theorem 2.7] we can realize $C^*(M)$ as the C*-algebra generated by $M$. Consequently we have the natural inclusions

$$J(M \otimes N) \subseteq C^*(M) \otimes C^*(N) \subseteq W^*(M) \otimes W^*(N).$$

Thus, it is clear that $JW(M \otimes N)$ is the weak-* closure of $J(M \otimes N)$ in $W^*(M) \otimes W^*(N)$. Using [20, 4.4.22] we have further inclusions

$$J(M \otimes N) \subseteq JC(M \otimes_{\min} N) \subseteq C^*(M) \otimes_{\min} C^*(N) \subseteq W^*(M) \otimes_{\min} W^*(N) \subseteq W^*(M) \otimes W^*(N)$$

where $W^*(M) \otimes_{\min} W^*(N)$ is of course weak-* dense in $W^*(M) \otimes W^*(N)$. Thus we also deduce that $JC(M \otimes_{\min} N)$ is weak-* dense in $JW(M \otimes N)$.

**Lemma 2.3.** Let $M \subseteq \mathcal{B}(H)_{s.a}$ and $N \subseteq \mathcal{B}(K)_{s.a}$ be JW-algebras. Then there is a normal Jordan homomorphism of $JW(M \otimes N)$ onto $JW_{H,K}(M \otimes N)$.

**Proof.** Let $\iota_M: M \rightarrow [M]^-$, and $\iota_N: N \rightarrow [N]^-$ be the inclusion maps. Then consider the surjective normal *-homomorphisms $\iota_M: W^*(M) \rightarrow [M]^-$, $\iota_N: W^*(N) \rightarrow [N]^-$ of $W^*(M) \otimes W^*(N)$ onto $[M]^- \otimes [N]^-$ and $[N]^-$ respectively. Let $\pi$ be the restriction of $\iota_M \otimes \iota_N$ to $JW(M \otimes N)$. Thus $\pi(JW(M \otimes N))$ is a JW-subalgebra of $([M]^- \otimes [N]^-)_{s.a}$, by [9, 4.5.5 and 4.5.11]. Since $M$ and $N$ generate $JW(M \otimes N)$ and $JW_{H,K}(M \otimes N)$ are of weak-* dense, proving the lemma.

The next result is apparent from Lemma 2.3 and [20, 4.5.2].

**Lemma 2.4.** Let $M$ and $N$ be JW-algebras. If $\pi_1: M \rightarrow \mathcal{B}(H)_{s.a}$ and $\pi_2: N \rightarrow \mathcal{B}(K)_{s.a}$ are faithful normal representations of $M$ and $N$ which extend to faithful normal representations $\hat{\pi}_1: W^*(M) \rightarrow \mathcal{B}(H)$, $\hat{\pi}_2: W^*(N) \rightarrow \mathcal{B}(K)$, then

$$JW(M \otimes N) \simeq JW_{H,K}(M \otimes N).$$

**Remark 2.5.** Let $M \subseteq \mathcal{B}(H)_{s.a}$ be a universally reversible JW-algebra with no non-zero weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra, then $[M]^- \simeq W^*(M)$. Indeed, if $R(M) \cap dR(M) = 0$, then the conclusion follows.
by Lemma 2.3. If $R(M)\cap iR(M) \neq 0$, then $J = R(M)^- \cap iR(M)^- \neq 0$, by [19, Lemma 2.3], and is a non-zero weakly-closed two sided ideal of $[M]^-$ (see [18, Remark 2.2]). In addition,

$$J_{s.a} = (R(M)^- \cap iR(M)^-)_{s.a} \subset R(M)_{s.a} = M,$$

by [2, p. 279]. Hence, $J_{s.a}$ is a non-zero weakly-closed Jordan ideal of $M$. This contradiction proves the assertion.

**Corollary 2.6.** Let $M \subset \mathcal{B}(H)_{s.a}$ and $N \subset \mathcal{B}(K)_{s.a}$ be universally reversible JW-algebras such that $M$ and $N$ contain no non-zero weakly-closed Jordan ideals isomorphic to the self-adjoint part of a Von Neumann algebra. Then

$$\mathrm{JW}(M \hat{\otimes} N) \simeq \mathrm{JW}_{H,H}(M \hat{\otimes} N).$$

It is easy to see that if $M$ is a JW-algebra, then $M$ is reversible in $W^*(M)$ if and only if $M$ is universally reversible. Thus, given a JC-algebra $A$, the separate weak *-continuity of multiplication together with the fact that $C^*(A)^* = W^*(A^*)$ [9, 7.1.11] imply that $A$ is universally reversible if and only if $A^*$ is universally reversible.

**Proposition 2.7.** Let $M$ and $N$ be JW-algebras. Then $\mathrm{JW}(M \hat{\otimes} N)$ is universally reversible unless one of $M$, $N$ has a non-zero abelian part and the other a non-zero Type $I_{\lambda,k}$ part, for some cardinal number (possibly infinite) $\geq 4$.

**Proof.** Suppose that the condition on $M$ and $N$ fails to occur. Then $\mathrm{JC}(M \otimes_{\min} N)$ is universally reversible, by [5, Corollary 1.5]. Hence $\mathrm{JW}(M \hat{\otimes} N)$ is universally reversible because it is the weak-closure of $\mathrm{JC}(M \otimes_{\min} N)$ in $W^*(M) \hat{\otimes} W^*(N)$.

On the other hand, suppose that $M$ has a non-zero abelian part $M_a \simeq C_R(X)$, say, where $X$ is a compact hyperstonean space and $N$ has a non-zero Type $I_{\lambda,k}$ part $M_k$, for some cardinal number $k \geq 4$. Then using Stacey's result [17], and Grothendieck's result [20] we see that, since $M_a \hat{\otimes} M_k$ has a weakly dense subalgebra isomorphic to $C_R(X) \otimes_{\alpha} (C(Y) \otimes_{\lambda} V_k)$, where $C_R(Y) \simeq Z(M_k)$ and $\lambda$ = least cross norm. $\mathrm{JW}(M \hat{\otimes} N)$ has a non-zero Type $I_{\lambda,k}$ part. Hence $\mathrm{JW}(M \hat{\otimes} N)$ is not universally reversible.

**Remark.** Corollary 2.6 cannot be improved, and as examples of failure:

(a) Let $M = N = V_5 (\simeq M_5(\mathcal{H}), s.a)$. By [3, p. 385]

$$\mathrm{JW}(M \hat{\otimes} N) \simeq \bigoplus^{4}_{i=1} M_{16}(\mathcal{R}), s.a.$$

However there is a natural embedding $\pi: V_5 \rightarrow \mathcal{B}(H)$, where $H = C^4$, in which case, $R(\pi(V_5)) \simeq M_5(\mathcal{H})$ and we see that

$$\mathrm{JW}_{H,H}(M \hat{\otimes} N) \simeq M_{16}(\mathcal{R}), s.a,$$

and hence, $\mathrm{JW}(M \hat{\otimes} N) \not\simeq \mathrm{JW}_{H,H}(M \hat{\otimes} N)$. 
(b) Put \( M = N = M_n(\mathbb{C})_{sa} \). We note that \( M_n(\mathbb{C})_{sa} \otimes M_n(\mathbb{C})_{sa} = M_n(\mathbb{C})_{sa} \). So from the natural identification \( M_n(\mathbb{C}) = \mathcal{B}(H) \), where \( H = \mathbb{C}^n \) we have

\[
JW_{H,H}(M \otimes N) = M_n(\mathbb{C})_{sa}.
\]

On the other hand, we have

\[
JW(M \otimes N) = \left( \frac{RW^*(M) \otimes RW^*(N)}{R} \right)_{sa}
\]

\[
= \left( M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \right)_{sa}
\]

\[
= \left( M_n(\mathbb{C})_{sa} + M_n(\mathbb{C})_{sa} \right)_{sa}.
\]

That is \( JW(M \otimes N) \neq JW_{H,H}(M \otimes N) \).

**Lemma 2.8.** Let \( M \subset \mathcal{B}(H)_{sa} \) be a universally reversible JW-algebra. Then \( RW^*(M) = R(M)^- \). Further, if \( R(M) \cap iR(M) = 0 \), then \( W^*(M) = [M]^\perp \).

**Proof.** The natural inclusion \( M \to \mathcal{B}(H)_{sa} \) gives rise to the real weak *-continuous surjection \( \pi: RW^*(M) \to R(M)^- \), where \( \pi \circ \psi_M(x) = x \), for all \( x \in M \), by the universal property. But note, \( x \in \text{Ker} \pi \) implies that \( x^*x \in RW^*(M)_{sa} = \psi_M(M) \), so that with \( \psi_M(y) = x^*x \) we have \( y = \pi \circ \psi_M(y) = \pi(x^*x) = 0 \). Hence \( x = 0 \), proving the first statement. The second statement is now immediate from the fact that \( R(M)^- \cap iR(M)^- = 0 \) [19, Lemma 2.3].

**Notation.** Let \( \mathcal{R} \) and \( \mathcal{S} \) be real subspaces of Von Neumann algebras \( \mathcal{M} \) and \( \mathcal{N} \), respectively. We denote by \( \mathcal{R} \otimes \mathcal{S} \) the real Von Neumann subalgebra of \( \mathcal{M} \otimes \mathcal{N} \) generated by the set \( \{ \sum_{j=1}^n r_j \otimes s_j : r_j \in \mathcal{R}, s_j \in \mathcal{S}, j = 1, \ldots, n \} \).

**Theorem 2.9.** Let \( M \) and \( N \) be JW-algebras, and \( \phi_M, \phi_N \) be the canonical *-antiautomorphisms of \( W^*(M) \), \( W^*(N) \), respectively. If \( JW(M \otimes N) \) is universally reversible, then

(i) \( W^*(JW(M \otimes N)) = W^*(M) \otimes W^*(N) \).

(ii) \( \phi_M \otimes \phi_N \) is the canonical *-antiautomorphism of \( W^*(JW(M \otimes N)) \), in which case, \( JW(M \otimes N) \) is exactly the self-adjoint fixed points of \( \phi_M \otimes \phi_N \).

**Proof.** Consider the *-antiautomorphism

\[
\phi_M \otimes \phi_N: W^*(M) \otimes W^*(N) \to W^*(M) \otimes W^*(N),
\]

and note that if \( x \in W^*(M) \), \( y \in W^*(N) \), then

\[
(\phi_M \otimes \phi_N)^2(x \otimes y) = \phi_M(x) \otimes \phi_N(y) = x \otimes y.
\]

Thus \( (\phi_M \otimes \phi_N)^2 \) is the identity map on \( W^*(M) \otimes W^*(N) \) since it is normal. Now, consider the real *-subalgebras, \( RW^*(M) = \{ x \in W^*(M) : \phi_M(x) = x^* \} \), \( RW^*(N) = \{ x \in W^*(N) : \phi_N(x) = x^* \} \) and \( R_\epsilon = \{ x \in W^*(M) \otimes W^*(N) : (\phi_M \otimes \phi_N)(x) = x^* \} \). Note that \( R_\epsilon \cap iR_\epsilon = 0 \), so that \( W^*(M) \otimes W^*(N) = R_\epsilon \oplus iR_\epsilon \), by [9, 7.3.2].
Clearly, $\mathcal{R}W^*(M) \otimes \mathcal{R}W^*(N)$ is the real Von Neumann subalgebra of $\mathcal{W}^*(M) \otimes \mathcal{W}^*(N)$, generated by $M \otimes N$, and therefore by $\mathcal{JW}(M \otimes N)$. Since $\mathcal{JW}(M \otimes N)$ is universally reversible, we have

$$\mathcal{R}W^*(\mathcal{JW}(M \otimes N)) = \mathcal{R}W^*(M) \otimes \mathcal{R}W^*(N),$$

by Lemma 2.8. If $x \in \mathcal{R}W^*(M)$, and $y \in \mathcal{R}W^*(N)$, then

$$(\phi_M \otimes \phi_N)(x \otimes y) = \phi_M(x) \otimes \phi_N(y) = x^* \otimes y^* = (x \otimes y)^*.$$ 

Thus, $\mathcal{R}W^*(M) \otimes \mathcal{R}W^*(N) \subseteq \mathcal{R}$, and so, $\mathcal{R}W^*(M) \otimes \mathcal{R}W^*(N) \subseteq \mathcal{R}$, since $\mathcal{R}$ is weak-* closed, by [9, 7.3.2]. Then from the above and Lemma 2.8 it follows that

$$\mathcal{R}W^*(\mathcal{JW}(M \otimes N)) = \mathcal{R}W^*(M) \otimes \mathcal{R}W^*(N) = \mathcal{R},$$

and that

$$\mathcal{W}^*(\mathcal{JW}(M \otimes N)) = \mathcal{W}^*(M) \otimes \mathcal{W}^*(N).$$

Part (ii) is now immediate from (i), and the universal reversibility of $\mathcal{JW}(M \otimes N)$.

Note that if $M$ and $N$ are $\mathcal{JW}$-algebras with no Type $I_2$ part, then $\mathcal{JW}(M \otimes N)$ is universally reversible and $\mathcal{W}^*(\mathcal{JW}(M \otimes N)) = \mathcal{W}^*(M) \otimes \mathcal{W}^*(N)$.

**Proposition 2.10.** Let $M$ be a universally reversible $\mathcal{JW}$-algebra with no abelian part. Then the following are equivalent:

(i) $M$ contains a weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra.

(ii) There is a non-zero projection $e$ in $Z(\mathcal{W}^*(M))$ such that $\phi_M(e) \neq e$.

(iii) $Z(M) \neq Z(\mathcal{W}^*(M))_{sa}$.

**Proof.**

(i) $\Rightarrow$ (ii). Suppose that $M$ contains a weakly-closed Jordan ideal $J$ isomorphic to the self-adjoint part of a Von Neumann algebra. By [9, 4.3.6], there exists a central projection $e$ in $M$ such that $J = eM$. Then $\mathcal{W}^*(eM) = e\mathcal{W}^*(M)$ is a weakly-closed ideal of $\mathcal{W}^*(M)$, and the restriction of $\phi_M$ to $\mathcal{W}^*(eM)$ is the canonical $*$-antiautomorphism of $\mathcal{W}^*(eM)$. Since $M$ has no abelian part, neither does $eM$. Therefore, there is, by [9, 7.4.7], a projection $f$ of $Z(\mathcal{W}^*(eM))$ and hence, of $Z(\mathcal{W}^*(M))$, such that $\phi_M(f) = e - f \neq f$.

(ii) $\Rightarrow$ (iii). Suppose that there is a projection $e$ in $Z(\mathcal{W}^*(M))$ such that $\phi_M(e) \neq e$. Since $M$ is universally reversible, it consists of all self-adjoint elements of $\mathcal{R}W^*(M)$, by [9, 7.3.3]. It follows that $e \notin M$, and hence $Z(M) \neq Z(\mathcal{W}^*(M))_{sa}$.

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Suppose that (iii) holds. Then $\phi_M(x) \neq x$, for some $x = x^* \notin Z(M)$. Hence there must be a projection $z$ in $Z(\mathcal{W}^*(M))$ such that $\phi_M(z) \neq z$, proving (ii). By [9, 7.3.5], there are projections $e \in Z(M), f \in Z(\mathcal{W}^*(M))$ such that $e + f + \phi_M(f) = 1$ and $(1 - e)M \simeq \mathcal{W}^*(M)_{sa}$. The condition (i) is immediate if $e \neq 1$. If $e = 1$, then $\phi_M$ leaves $Z(\mathcal{W}^*(M))$ pointwise invariant by [9, 4.2.15], a contradiction.

**Lemma 2.11.** Let $M$ be a $\mathcal{JW}$-algebra. If there exists a weakly closed ideal $J$ of $\mathcal{W}^*(M)$ such that $\mathcal{W}^*(M) = J \oplus \phi_M(J)$, then $J_{sa} \simeq \mathcal{R}W^*(M)_{sa} = M$ via $x \mapsto x \oplus \phi_M(x)$.

**Proof.** The proof is exactly the same as in Lemma 1.1. (ii) $\Rightarrow$ (i) of [6].
THEOREM 2.12. Let $M$ be a Von Neumann algebra with no abelian part, and let $N$ be any JW-algebra. Then

$$\text{JW}(M \otimes N) \simeq (M \otimes W^*(N))_{sa}.$$  

PROOF. Put $M = M_{sa}$, then $W^*(M) = M \oplus M^0$, where the inclusion map of $M$ into $W^*(M)$ is given by $a \mapsto a \oplus a^0$, and the canonical $*$-antiautomorphism $\phi_M$ of $W^*(M)$ is defined by $\phi_M(a \oplus b^0) = b \oplus a^0$, by [9, 7.4.7]. Since $M$ is a universally reversible JW-algebra, by [9, 7.4.6], which has no one-dimensional representations, $\text{JW}(M \otimes N)$ is universally reversible, by Proposition 2.7, and so

$$W^*(\text{JW}(M \otimes N)) = W^*(M) \otimes W^*(N),$$

$$= (M \oplus M^0) \otimes W^*(N),$$

$$= (M \otimes W^*(N)) \oplus (M^0 \otimes W^*(N)).$$

Since $\phi_M(M) = M^0$, we have $(\phi_M \otimes \phi_N)(M \otimes W^*(N)) = M^0 \otimes W^*(N)$. But $\phi_M \otimes \phi_N$ is normal on $W^*(M) \otimes W^*(N)$. Hence, it is normal on $M \otimes W^*(N)$, and so $(\phi_M \otimes \phi_N)(M \otimes W^*(N)) = M^0 \otimes W^*(N)$. That is

$$W^*(\text{JW}(M \otimes N)) = (M \otimes W^*(N)) \oplus (\phi_M \otimes \phi_N)(M \otimes W^*(N)).$$

The described conclusion now follows from Lemma 2.11 because $\phi_M \otimes \phi_N$ is the canonical $*$-antiautomorphism of $W^*(\text{JW}(M \otimes N))$.

REMARK 2.13. Let $M_1$, $M_2$, $N$ be JW-algebras. Then

$$\text{JW}((M_1 \oplus M_2) \otimes N) = \text{JW}(M_1 \otimes N) \oplus \text{JW}(M_2 \otimes N).$$

This follows easily from the definitions because

$$W^*(M_1 \oplus M_2) \otimes W^*(N) = (W^*(M_1) \oplus W^*(M_2)) \otimes W^*(N),$$

$$= (W^*(M_1) \otimes W^*(N)) \oplus (W^*(M_2) \otimes W^*(N)).$$

THEOREM 2.14. Let $M \subseteq \mathcal{B}(H)$, $N \subseteq \mathcal{B}(K)$ be universally reversible JW-algebras with no abelian part. Then the following are equivalent:

(i) $\text{JW}(M \otimes N)$ contains no non-zero weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra.

(ii) Neither $M$ nor $N$ contains a non-zero weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra.

(iii) $\text{JW}_{H,K}(M \otimes N)$ contains no non-zero weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra.

If one (and hence all) of these conditions is satisfied, then

$$\text{JW}(M \otimes N) \simeq \text{JW}_{H,K}(M \otimes N).$$
PROOF. (i) $\Rightarrow$ (ii). Suppose that (i) holds, and assume that $M$ contains a non-zero weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra. Then by Remark 2.13 and Theorem 2.12 we see that $JW(M \widehat{\otimes} N)$ contains a weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra. Hence (ii) follows by contradiction.

(ii) $\Rightarrow$ (i). If (ii) holds, then $\phi_M$ and $\phi_N$ fix the centres of $W^*(M)$ and $W^*(N)$, respectively, by Proposition 2.10. Since $M$ and $N$ are universally reversible, $W^*(JW(M \widehat{\otimes} N)) = W^*(M) \widehat{\otimes} W^*(N)$, and so

$$Z\left(W^*(JW(M \widehat{\otimes} N))\right) = Z\left(W^*(M) \widehat{\otimes} W^*(N)\right)$$

by [20, 4.5.11]. Since $\phi_M \widehat{\otimes} \phi_N$ is the canonical *-antiautomorphism of $W^*(M) \widehat{\otimes} W^*(N)$ and, by the above, fixes the centre of $W^*(M) \widehat{\otimes} W^*(N)$, the conclusion follows from Proposition 2.10.

(i) $\Rightarrow$ (iii). By Lemma 2.3, $JW_{H,K}(M \widehat{\otimes} N)$ is isomorphic to a weakly-closed Jordan ideal of $\text{JW}(M \widehat{\otimes} N)$.

(iii) $\Rightarrow$ (ii). Let $I$ be a weakly-closed Jordan ideal of $M$ such that $I$ is isomorphic to the self-adjoint part of a Von Neumann algebra. Then $JW_{H,K}(I \widehat{\otimes} N)$ is a weakly-closed Jordan ideal of $JW_{H,K}(M \widehat{\otimes} N)$, and is isomorphic to an ideal of $\text{JW}(I \widehat{\otimes} N)$, by Lemma 2.3. But the latter is isomorphic the self-adjoint part of a Von Neumann algebra, by Theorem 2.12.

The final statement follows from Corollary 2.6.

3. On the centre of the tensor product of JW-algebras. It is well-known (see 20, [4.5.11]) that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are Von Neumann algebras with centres $Z_1$ and $Z_2$, respectively, then the centre of the tensor product $\mathcal{M}_1 \widehat{\otimes} \mathcal{M}_2$ is $Z_1 \widehat{\otimes} Z_2$. It is natural to ask whether a similar result holds in the context of JW-algebras, and the JW-tensor product.

It transpires that the answer is "no" in general, but that the situation is manageable if we confine ourselves to universally reversible JW-algebras.

The following examples give some idea of the difficulties.

EXAMPLE 3.1. Let $M = N = \mathcal{B}(H)_{sa}$, where $H$ is a complex Hilbert space. Then $JW(M \widehat{\otimes} N) \simeq \left(\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)\right)_{sa} \oplus \left(\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)\right)_{sa} \simeq \mathcal{B}(H \otimes H)_{sa} \oplus \mathcal{B}(H \otimes H)_{sa}$ so that

$$Z(JW(M \widehat{\otimes} N)) = \mathbb{R} \oplus \mathbb{R} \neq \mathbb{R} = Z(M) \widehat{\otimes} Z(N).$$

EXAMPLE 3.2. Let $M = N = V_{4n+1}$, for some $n < \infty$. Then using [6, Theorem 8] we have

$$Z(JW(M \widehat{\otimes} N)) = Z(JC(V_{4n+1} \otimes V_{4n+1}))$$

$$= \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \neq \mathbb{R}$$

$$= Z(M) \widehat{\otimes} Z(N).$$
EXAMPLE 3.3. Let $M = N = V$ be an infinite dimensional spin factor. Then

$$\text{JW}(M \otimes N) = (\text{RW}^*(V) \otimes \text{RW}^*(V))_{s.a},$$

by Theorem 2.9, and Proposition 2.7. In this case, again $Z(M) \otimes Z(N) = R$. But $Z(\text{JW}(M \otimes N))$ is infinite dimensional. This is because $Z(\text{RW}^*(V))$ is infinite dimensional. Indeed, $W^*(V) = W^*(V)** = C^*(V)**$ by [9, 6.1.7(iv) and 7.1.11] and $C^*(V)$ has an uncountable family of inequivalent irreducible representations [13, Section 6]. So, $C^*(V)**$ certainly has an infinite orthogonal family of minimal central projections, and, since for any projection $e \in C^*(V)**$ we have $e + \phi_V(e) \in \text{RW}^*(V)$, it follows that $\text{RW}^*(V)$ must have an infinite dimensional centre.

In view of the intractability, in this context, of the Type $I_2$ JW-algebras (described above) we will concentrate on the cases where both $M$ and $N$ are universally reversible. Even though it is not always (even then) the case that $Z(\text{JW}(M \otimes N)) = Z(M) \otimes Z(N)$, we can, nevertheless, describe $Z(\text{JW}(M \otimes N))$ in terms of $Z(M) \otimes Z(N)$ (Theorem 3.10).

PROPOSITION 3.4. Let $M$ and $N$ be universally reversible JW-algebras such that each of the centres of $W^*(M)$ and $W^*(N)$ are pointwise fixed by their respective canonical $*$-antiautomorphism. Then

$$Z(\text{JW}(M \otimes N)) = Z(M) \otimes Z(N).$$

PROOF. By [9, 7.3.3] $Z(W^*(M))_{s.a} = Z(M)$, and $Z(W^*(N))_{s.a} = Z(N)$. Now the fact that $W^*(\text{JW}(M \otimes N)) = W^*(M) \otimes W^*(N)$, and $Z(\text{JW}^*(M) \otimes \text{W}^*(N)) = Z(W^*(M) \otimes W^*(N))$ together with Lemma 2 of [14] implies that

$$Z\left(W^*(\text{JW}(M \otimes N))\right)_{s.a} = Z(M) \otimes Z(N).$$

But since $Z(W^*(M)), Z(W^*(N))$ are pointwise invariant under $\phi_M, \phi_N$ respectively, and $\phi_M \otimes \phi_N$ is normal on $W^*(M) \otimes W^*(N)$, it follows that $Z(W^*(M) \otimes Z(W^*(N))$ is pointwise $\phi_M \otimes \phi_N$-invariant, from which it follows that

$$Z(\text{JW}(M \otimes N)) = Z(W^*(\text{JW}(M \otimes N)))_{s.a},$$

$$= Z(M) \otimes Z(N),$$

proving the proposition.

COROLLARY 3.5. Let $M$ and $N$ be universally reversible JW-algebras such that neither contains a non-zero Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra. Then $Z(\text{JW}(M \otimes N)) = Z(M) \otimes Z(N)$.

PROOF. This is immediate from Proposition 3.4, and Proposition 2.10.
LEMMA 3.6. Let \( \mathcal{M} \) be a Von Neumann algebra without abelian part, and \( N \) a JW-algebra. Then
\[
Z(\mathcal{JW}(\mathcal{M}_{sa} \hat{\otimes} N)) \simeq Z(\mathcal{M})_{sa} \hat{\otimes} Z(\mathcal{W}^*(N))_{sa}.
\]

PROOF. Note first that for any Von Neumann algebra \( W \), \( Z(W_{sa}) = Z(W)_{sa} \). By Theorem 2.12, \( \mathcal{JW}(\mathcal{M}_{sa} \hat{\otimes} N) \simeq (\mathcal{M} \hat{\otimes} \mathcal{W}^*(N))_{sa} \). So by [20, 4.5.11] and [14, Lemma 2] we have
\[
Z(\mathcal{JW}(\mathcal{M}_{sa} \hat{\otimes} N)) \simeq Z(\mathcal{M} \hat{\otimes} \mathcal{W}^*(N))_{sa} = \left( Z(\mathcal{M}) \hat{\otimes} Z(\mathcal{W}^*(N)) \right)_{sa} = Z(\mathcal{M})_{sa} \hat{\otimes} Z(\mathcal{W}^*(N))_{sa},
\]
and the proof is complete.

LEMMA 3.7. Let \( \mathcal{M} \) and \( \mathcal{N} \) be Von Neumann algebras without abelian part. Then
\[
Z(\mathcal{JW}(\mathcal{M}_{sa} \hat{\otimes} \mathcal{N}_{sa})) \simeq (Z(\mathcal{M})_{sa} \hat{\otimes} Z(\mathcal{N})_{sa}) \oplus (Z(\mathcal{M})_{sa} \hat{\otimes} Z(\mathcal{N})_{sa}).
\]

PROOF. This is immediate since
\[
\mathcal{JW}(\mathcal{M}_{sa} \hat{\otimes} \mathcal{N}_{sa}) \simeq (\mathcal{M} \hat{\otimes} \mathcal{N})_{sa} \oplus (\mathcal{M}^0 \hat{\otimes} \mathcal{N}^0)_{sa}.
\]

COROLLARY 3.8. Let \( M \) be an abelian JW-algebra and \( N \) any universally reversible JW-algebra. Then
\[
Z(\mathcal{JW}(M_{sa} \hat{\otimes} N)) = M \hat{\otimes} Z(N).
\]

PROOF. Observe that \( \mathcal{JW}(M \hat{\otimes} N) = M \hat{\otimes} N \), and that \( M = M_{sa} \) for some abelian Von Neumann algebra \( M \). By Lemma [9, 7.3.5] we can write \( N = A \oplus W_{sa} \), where the centre of \( \mathcal{W}^*(A) \) is pointwise fixed by \( \phi_N \) and, \( W \) is a Von Neumann algebra. Then
\[
Z(\mathcal{JW}(M \hat{\otimes} A)) = M \hat{\otimes} Z(A),
\]
by Proposition 3.4. Also, using [20, 4.5.11],
\[
Z(\mathcal{JW}(M_{sa} \hat{\otimes} W_{sa})) = Z(M_{sa} \hat{\otimes} Z(W)_{sa}) = M \hat{\otimes} Z(W_{sa}).
\]
Hence,
\[
Z(\mathcal{JW}(M \hat{\otimes} N)) = M \hat{\otimes} Z(N).
\]

REMARK. According to Proposition 2.10, a given universally reversible JW-algebra \( M \) has a decomposition
\[
M = M_1 \oplus M_2 \oplus M_3,
\]
where \( M_1 \) is the abelian part of \( M \), the centre of \( \mathcal{W}^*(M_1 \oplus M_2) \) is pointwise fixed by its canonical *-antiautomorphism and, \( M_3 \) is isomorphic to the self-adjoint part of a Von Neumann algebra.

Retaining this notation, we prove:
Theorem 3.10. Let $M$ and $N$ be universally reversible JW-algebras. Then

$$Z(JW(M \otimes N)) \simeq (Z(M) \hat{\otimes} Z(N)) \oplus (Z(M_1) \hat{\otimes} Z(N_2)).$$

Proof. By Remark 2.13, we have

$$JW(M \otimes N) = JW((M_1 \oplus M_2) \hat{\otimes} (N_1 \oplus N_2)) \oplus JW((M_1 \oplus M_2) \hat{\otimes} N_1)$$

$$\oplus JW(M_2 \hat{\otimes} (N_1 \oplus N_2)) \oplus JW(M_2 \hat{\otimes} N_1).$$

Now,

$$Z(JW((M_1 \oplus M_2) \hat{\otimes} (N_1 \oplus N_2))) = Z(M_1 \oplus M_2) \hat{\otimes} Z(N_1 \oplus N_2),$$

by Proposition 3.4. By Lemma 3.6, we get

$$Z(JW(M_1 \hat{\otimes} M_2)) = (Z(M_1) \hat{\otimes} Z(N_1)) \oplus (Z(M_1) \hat{\otimes} Z(N_2)).$$

Similarly,

$$Z(JW(M_2 \hat{\otimes} (N_1 \oplus N_2))) = Z(M_2) \hat{\otimes} Z(N_1 \oplus N_2).$$

By Corollary 3.7, we get

$$Z(JW(M \hat{\otimes} N)) = (Z(M) \hat{\otimes} Z(N)) \oplus (Z(M_1) \hat{\otimes} Z(N_2)).$$

Collecting terms, we have

$$Z(JW(M \otimes N)) \simeq (Z(M) \hat{\otimes} Z(N)) \oplus (Z(M_1) \hat{\otimes} Z(N_2)).$$

as required.

4. The type of the tensor products of JW-algebras. In this section we investigate the type of $JW(M \otimes N)$, where $M$ and $N$ are JW-algebras. We first study the type of $JW(M \otimes N)$ when $M$ and $N$ are of Type I. Then we complete the discussion of tensoring different types of JW-algebras. In one sense, the work can be considered as an application of Theorem 2.9, Theorem 8 of [1] and Table 11.1 of [11]. However, our main object is to establish the “multiplication table” (Theorem 4.9) which describes completely the type of the tensor product $JW(M \otimes N)$.

Lemma 4.1. Let $M$ be a JW-algebra, and let $N$ be a Type I, JW-factor, $n < \infty$, and $X$ any compact Hausdorff space. Suppose that $M$ contains a weak-* dense JC-subalgebra of the form $C(X, N)$. Then $M$ is the same type as $N$.

Proof. If $N$ is an infinite dimensional spin factor, then this follows from the result of Stacey [17]. So we may suppose that $N$ is any finite dimensional factor. By [7, Corollary 5.1] for example, the fact that $N$ is finite dimensional implies that $C(X, N)'' \simeq C(X)'' \otimes N \simeq C(Y) \otimes N$, for some compact hyperstonean space $Y$. But by hypothesis, there is a surjective normal homomorphism $C(X, N)'' \rightarrow M$ and the described conclusion follows immediately from this.
LEMMA 4.2. Let $A, B$ be non-abelian finite dimensional JW-factors, and let $X, Y$ be compact Hausdorff spaces. Then

$$JC\left( (C(X, A) \otimes_{\min} C(Y, B)) \right) = C(X \times Y) \otimes \left( R^*(A) \otimes_R R^*(B) \right)_{s.a}.$$

PROOF. $JC\left( (C(X, A) \otimes_{\min} C(Y, B)) \right)$ is universally reversible, by [4, Corollary 1.5]. Thus we see that, using [4, Theorem 2.2]

$$C_R(X) \otimes_R C_R(Y) \otimes_R \left( R^*(A) \otimes_R R^*(B) \right)_{s.a} = \left( C_R(X) \otimes_R R^*(A) \otimes_R C_R(Y) \otimes_R R^*(B) \right)_{s.a}$$

is norm dense in $JC\left( (C(X, A) \otimes_{\min} C(Y, B)) \right)$, and so,

$$JC\left( (C(X, A) \otimes_{\min} C(Y, B)) \right) = C(X \times Y) \otimes \left( R^*(A) \otimes_R R^*(B) \right)_{s.a},$$

as required.

4.3. From [12, Lemma 3.1.7] we have the following real algebra identifications:

$$C \otimes_R C = C \oplus C, \quad C \otimes_R H = M_2(C), \quad H \otimes_R H = M_4(R),$$

and consequently the following tensor product table for $M_n(F) \otimes M_m(F')$

<table>
<thead>
<tr>
<th>$M_n(F)$</th>
<th>$M_m(C)$</th>
<th>$M_m(H)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_n(R)$</td>
<td>$M_m(R)$</td>
<td>$M_m(C)$</td>
</tr>
<tr>
<td>$M_n(C)$</td>
<td>$M_m(C)$</td>
<td>$M_m(C) \oplus M_m(C)$</td>
</tr>
<tr>
<td>$M_n(H)$</td>
<td>$M_m(H)$</td>
<td>$M_{2m}(C)$</td>
</tr>
<tr>
<td>$M_m(R)$</td>
<td>$M_m(C)$</td>
<td>$M_{4m}(R)$</td>
</tr>
</tbody>
</table>

Recall that the Type $I_n$ factors, $n < \infty$, are $M_n(F)_{s.a}$, where $F = R$, $C$ or $H$ and the spin factors [9, 5.3.8]. Also, recall that $R^*(M_n(F))_{s.a} = M_n(F)$, whenever $n \geq 3$ and $F = R$, $C$ or $H$ [10, Corollary 1, p. 143]. Thus, using 4.1–4.3 together with the fact that $JC(M \otimes N)$ is weak $\ast$-dense in the JW-tensor product $JW(M \otimes N)$ we obtain the following, which is a complete description of $JW(M \otimes N)$, where $M$ and $N$ are JW-algebras of Type $I_m$ with no Type $I_{2,\infty}$ part.

THEOREM 4.4. We have the following tensor product table for $JW(M \otimes N)$

(i) $I_n F \otimes I_m F'$ $(3 \leq n, m < \infty, F, F' = R, C, H)$

<table>
<thead>
<tr>
<th>$I_n R$</th>
<th>$I_m R$</th>
<th>$I_m C$</th>
<th>$I_m H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_n R$</td>
<td>$I_m R$</td>
<td>$I_m R$</td>
<td>$I_m R$</td>
</tr>
<tr>
<td>$I_n C$</td>
<td>$I_m C$</td>
<td>$I_m C$</td>
<td>$I_{2m} C$</td>
</tr>
<tr>
<td>$I_n H$</td>
<td>$I_m H$</td>
<td>$I_m R$</td>
<td>$I_{4m} R$</td>
</tr>
</tbody>
</table>
(ii) $I_{2k} \otimes I_{2k}$ \((3 \leq n < \infty, F = R, C, H; k = 2, \ldots, 9)\)

<table>
<thead>
<tr>
<th>$I_{2k}$</th>
<th>$I_{2,2}$</th>
<th>$I_{2,3}$</th>
<th>$I_{2,4}$</th>
<th>$I_{2,5}$</th>
<th>$I_{2,6}$</th>
<th>$I_{2,7}$</th>
<th>$I_{2,8}$</th>
<th>$I_{2,9}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{2kR}$</td>
<td>$I_{2,2kR}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kN}$</td>
<td>$I_{2,2kN}$</td>
<td>$I_{2,2kN}$</td>
<td>$I_{2,2kN}$</td>
<td>$I_{2,2kN}$</td>
</tr>
<tr>
<td>$I_{2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
</tr>
<tr>
<td>$I_{2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
</tr>
</tbody>
</table>

(iii) $I_{2k} \otimes I_{2k'}$ \((k, k' = 2, \ldots, 9)\)

<table>
<thead>
<tr>
<th>$I_{2k}$</th>
<th>$I_{2,2}$</th>
<th>$I_{2,3}$</th>
<th>$I_{2,4}$</th>
<th>$I_{2,5}$</th>
<th>$I_{2,6}$</th>
<th>$I_{2,7}$</th>
<th>$I_{2,8}$</th>
<th>$I_{2,9}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{2kR}$</td>
<td>$I_{2,2kR}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kN}$</td>
<td>$I_{2,2kN}$</td>
<td>$I_{2,2kN}$</td>
<td>$I_{2,2kN}$</td>
<td>$I_{2,2kN}$</td>
</tr>
<tr>
<td>$I_{2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
<td>$I_{2,2kC}$</td>
</tr>
<tr>
<td>$I_{2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
<td>$I_{2,2kH}$</td>
</tr>
</tbody>
</table>

**Corollary 4.5.** If $M$ and $N$ are JW-algebras of Type $I_{\text{fin}}$ without Type $I_{2,oo}$ part then $JW(M \otimes N)$ is of the same type (without Type $I_{2,oo}$ part).

In order to complete the description of $JW(M \otimes N)$ for the remaining types of $M$ and $N$ we will need results which relate the type of $M$ to that of $W^*(M)$.

**Lemma 4.6.** Let $V$ be an infinite dimensional spin factor. Then $W^*(V)$ has no Type $I_{\text{fin}}$ part.

**Proof.** On the contrary, suppose that $W^*(V)$ does have (non-zero) Type $I_{\text{fin}}$ part. Then there is a central projection $e$ of $W^*(V)$ such that $eW^*(V) \simeq C(X, M_n(C))$ for some $n < \infty$ and some compact hyperstonean space $X$ (see [9, 7.4.5]). Then $0 \neq eV \subset C(X, M_n(C))$. Also $V \simeq eV$ because $V$ is simple. Therefore, the postliminal $C^*$-algebra $C(X, M_n(C))$ contains a copy of the antiliminal Clifford $C^*$-algebra $C^*(V)$. This is impossible, by [13, Proposition 6.2.9].

**Theorem 4.7.** Let $M$ be a JW-algebra. Then

(a) $M$ is of Type $I_{\text{fin}}$ without Type $I_{2,oo}$ part if and only if $W^*(M)$ is of Type $I_{\text{fin}}$.

(b) if $M$ is universally reversible then $M$ is of Type $I_{oo}$ (resp. $II_1$, $II_{oo}$, $III$) if and only if $W^*(M)$ is of Type $I_{oo}$ (resp. $II_1$, $II_{oo}$, $III$).

(We note that $M$ is universally reversible if it is of Type $I_{oo}$, $II_1$, $II_{oo}$ or $III$).

**Proof.** (b) This is due to Ajupov [1, Theorem 8] and Størmer [19, Corollary 6.5].
(a) Suppose that $W^*(M)$ is of Type $I_{0\infty}$. Then $M$ is certainly finite because any family of orthogonal equivalent projections of $M$ are orthogonal and unitarily equivalent in $W^*(M)$ and so must be finite. Now (b) implies that $M$ must be of Type $I_{0\infty}$. If $M$ has a (non-zero) Type $I_{0\infty}$ then it must contain a copy of an infinite dimensional spin factor $V$. But then $[V]^{-}$, the $W^*$-algebra generated in $W^*(M)$ by $V$ is not of Type $I_{0\infty}$, by Lemma 4.6, because it is a quotient of $W^*(V)$. Since every $W^*$-subalgebra of a Type $I_{0\infty}$ $W^*$-algebra is also of Type $I_{0\infty}$, this is a contradiction.

By (b), and in view of [4, Lemma 2.6], in order to prove the converse it is enough to suppose that $M = C(X, V_k)$, where $k < \infty$ and $X$ is compact hyperstonean. But then (see Lemma 2.5 of [4], $W^*(M) = C(X, W^*(V_k))$ which is of Type $I_{0\infty}$, and the proof is complete.

**Theorem 4.9.** The type of $JW(M \otimes N)$ for $JW$-algebras $M$ and $N$ is given in the following table

<table>
<thead>
<tr>
<th>$I_{2, k &lt; \infty}$</th>
<th>$I_{\infty}$</th>
<th>$I_{1}$</th>
<th>$I_{\infty}$</th>
<th>$III$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{\infty}$</td>
<td>$I_{\infty}$</td>
<td>$I_{1}$</td>
<td>$I_{\infty}$</td>
<td>$III$</td>
</tr>
<tr>
<td>$I_{1}$</td>
<td>$I_{1}$</td>
<td>$I_{\infty}$</td>
<td>$I_{1}$</td>
<td>$III$</td>
</tr>
<tr>
<td>$I_{\infty}$</td>
<td>$I_{\infty}$</td>
<td>$I_{1}$</td>
<td>$I_{\infty}$</td>
<td>$III$</td>
</tr>
<tr>
<td>$III$</td>
<td>$III$</td>
<td>$III$</td>
<td>$III$</td>
<td>$III$</td>
</tr>
</tbody>
</table>

**Proof.** Let $M$ and $N$ be any $JW$-algebra of any of the types occurring in the table. By Theorem 4.7 $M$ and $W^*(M)$ (respectively, $N$ and $W^*(N)$) are of the same type. In addition $JW(M \otimes N)$ and $W^*(M) \otimes W^*(N)$ are of the same type. This is because, by Proposition 2.7 and Theorem 2.9, $JW(M \otimes N)$ is universally reversible with $W^*(JW(M \otimes N)) = W^*(M) \otimes W^*(N)$, and so the claim follows from Theorem 4.7. The above table is now an immediate consequence of [15, Theorem 2.6.6], [20, Theorem 5.2.30].

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**References**


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