

# Influence Line Analysis of Bridges Using MATLAB

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## ABSTRACT

This paper presents a computer procedure for computing influence lines for plane frame bridge models. The procedure is based on the stiffness matrix equation in conjunction with fixed end actions representing the effect of moving load. The procedure does not need subdivision of the structure members or specification of load position prior to stiffness matrix analysis. Specification of the load and effect positions comes after the stiffness matrix analysis. The procedure was implemented into a computer program using MATLAB and can easily be implemented into any standard computer program for matrix structural analysis.

## المخلص

هذا البحث يقدم طريقة جديدة وفعالة لحساب خطوط التأثير للجسور ذات المنشآت غير المحددة. تعتمد هذه الطريقة على استخدام التحليل الإنشائي بالمصفوفات مع تمثيل الحمل المتحرك بتأثيرات مكافئة عند أطراف الأعضاء. وتكمن فعالية هذه الطريقة في عدم الحاجة لتقسيم أعضاء المنشأ إلى أجزاء أو تحديد معدل تحرك القوة أو خطوط التأثير المطلوبة قبل التحليل بالمصفوفات. جميع خطوط التأثير المطلوبة يتم الحصول عليها، بعد تحليل المنشأ الأساسي بالمصفوفات، بالطرق الأستاتيكية المعروفة. الطريقة سهلة ويمكن تطبيقها في أي برنامج كمبيوتر للتحليل بالمصفوفات وفي هذا البحث تم برمجة الطريقة باستخدام برنامج MATLAB.

## INTRODUCTION

Influence lines have important applications for the design of structures that are subjected to live or moving loads, e.g. bridges. Influence lines show the influence of positioning of a unit load on a function [1,2]. The function (or load effect) could be anything that varies as the load traverse the structure, such as moment or shear at a point in a structure, axial force in a member or support

reaction. Once the influence line for an effect at a point in a structure has been established, the maximum effect caused by the live load can be determined.

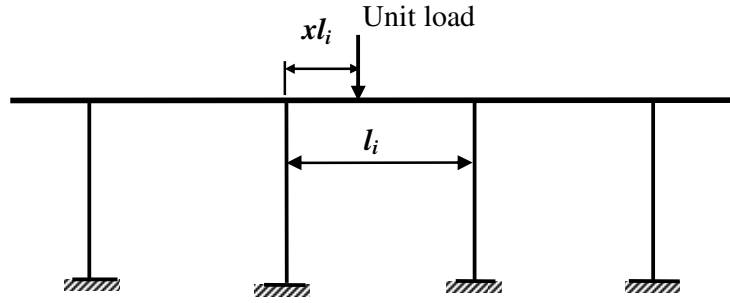
Although determination of influence lines for determinate structures is straightforward, it is rather involved for indeterminate structures. Obviously, determination of influence lines by positioning unit loads at several locations on the structure is impractical. Belegundu [3] presented a procedure for computing influence lines for determinate and indeterminate structures based on what he called the 'adjoint variable method'. The method, however, requires subdivision of the members in order to calculate the influence lines for functions within members. Furthermore, the effects for which influence lines are desired need to be specified prior to finite element analysis. The Muller-Breslau principle, on the other hand, provides a more effective way of obtaining qualitative influence lines. For quantitative evaluation of influence lines, the Muller-Breslau procedure is somewhat involved. Cifuentes and Paz [4], however, utilized the Muller-Breslau principle in conjunction with finite elements to determine the influence lines. They used the NASTRAN commercial code to demonstrate the procedure. The procedure, however, requires subdivision of the structure members in order to calculate the influence lines at points within members. Moreover, for each influence line, a new input data and hence a new round of finite element analysis is needed. Bridge oriented software like *QconBridge* [5] are based on subdivision of bridge spans into segments, normally ten segments per span, prior to finite element analysis. This results in a higher degree of freedom model and hence expensive computation cost. The method presented in this paper is based on the stiffness method and has the advantage of not requiring subdivision of members nor the specification of the desired load effect, its location or the load movement increment prior to the stiffness matrix analysis. After a standard stiffness matrix analysis has been completed, the method relies on simple statics to calculate the influence lines for any effect at any section in the structure using any desired load movement increment.

## COMPUTATIONAL PROCEDURE

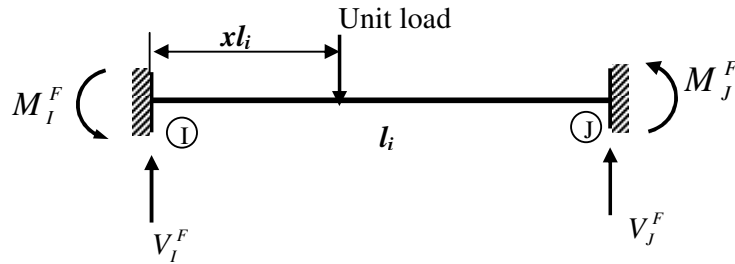
Consider the frame shown in Fig. 1. The stiffness matrix equation for the frame can be written as [6]

$$[\mathbf{K}]\Delta = \mathbf{F} - \mathbf{F}^F \quad (1)$$

where  $[\mathbf{K}]$  is  $n \times n$  stiffness matrix of the structure,  $\Delta$  is an  $(n \times 1)$  vector of nodal displacements and  $\mathbf{F}$  and  $\mathbf{F}^F$  are, respectively, the vectors of nodal and fixed-end forces associated with member loadings. Since there are no applied nodal forces, the vector  $\mathbf{F}$  contains zeros for all degrees of freedom except those associated with support reactions.



(a)



(b)

Figure 1: (a) Plane frame with a unit load on member  $i$ . (b) Fixed-end forces of member  $i$ .

**Fixed-End Forces:** The Fixed-end forces due to a unit transverse load positioned at distance  $xl_i$  ( $0 \leq x \leq 1$ ) from the left node (I-node) of span  $i$  can be written as

$$\mathbf{f}_i^F(x) \equiv \begin{Bmatrix} N_I^F \\ V_I^F \\ M_I^F \\ N_J^F \\ V_J^F \\ M_J^F \end{Bmatrix}_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 2 \\ 0 & l_i & -2l_i & l_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -l_i & l_i \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{Bmatrix} \quad (2a)$$

or symbolically

$$\mathbf{f}_i^F(x) = [\mathbf{h}]_i \mathbf{g}(x) \quad (2b)$$

where

$$\mathbf{g}(x) = \{1 \quad x \quad x^2 \quad x^3\}^T \quad (3)$$

Thus, the *vector function* of fixed-end forces can be regarded as a ‘nonlinear’ combination of four *constant vectors* (the vectors of  $[\mathbf{h}]$ ) that are independent of load position,  $x$ . The combination coefficients are the terms of  $\mathbf{g}(x)$ .

The stiffness matrix equation for member  $i$  can then be expressed as

$$[\mathbf{k}]_i \Delta(x) = \mathbf{f}(x) - [\mathbf{h}]_i \mathbf{g}(x) \quad (4)$$

where  $\delta(x)$  and  $\mathbf{f}(x)$  are, respectively, vector functions of the displacements and forces at the member ends.

Equation (4) can be transformed to global coordinates to yield

$$[\mathbf{K}]_i \Delta(x) = \mathbf{F}(x) - [\mathbf{H}]_i \mathbf{g}(x) \quad (5)$$

where

$$\begin{aligned} [\mathbf{K}]_i &= [\mathbf{C}]^T [\mathbf{k}]_i [\mathbf{C}], \\ \Delta(x) &= [\mathbf{C}]^T \delta(x), \\ \mathbf{F}(x) &= [\mathbf{C}]^T \mathbf{f}(x), \\ [\mathbf{H}]_i &= [\mathbf{C}]^T [\mathbf{h}]_i \end{aligned} \quad (6)$$

and  $[\mathbf{C}]$  is the displacement transformation matrix for a plane frame member [6]. Equations similar to eq. (5) can be obtained for all spans that would be subjected to the moving load. For other members the fixed-end force terms in eq.(5) vanish.

Now let  $n$  be the total degrees of freedom of the structure and  $m$  the number of spans that would be loaded, then the stiffness matrix equation of the structure can be assembled from the member stiffness equations (eq. 6) to yield the following

$$[\mathbf{K}] \Delta(x) = \mathbf{F}(x) - \left[ [\bar{\mathbf{H}}]_1 \mid [\bar{\mathbf{H}}]_2 \mid \cdots \mid [\bar{\mathbf{H}}]_m \right]_{(n \times 4m)} \begin{Bmatrix} \mathbf{g}_1(x) \\ \mathbf{g}_2(x) \\ \vdots \\ \mathbf{g}_m(x) \end{Bmatrix}_{(4m \times 1)} \quad (7)$$

where  $[\bar{\mathbf{H}}]_i$  is an  $n \times 4$  matrix whose rows contain zeros and the rows of the *six-row* matrix  $[\mathbf{H}]_i$ , arranged according to the global coordinate numbers of member  $i$  and

$$\mathbf{g}_i(x) = \begin{cases} \mathbf{g}(x) & \text{when the unit load is on member } i, \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Noting that the unknown nodal displacements in the vector  $\Delta(x)$  correspond to zero forces in the vector  $\mathbf{F}(x)$ , eq.(7) can be partitioned as follows:

$$\begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fp} \\ \mathbf{K}_{pf} & \mathbf{K}_{pp} \end{bmatrix} \begin{Bmatrix} \Delta_f(x) \\ \Delta_p \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{F}_p(x) \end{Bmatrix} - \left[ \begin{array}{c|c|c|c} [\bar{\mathbf{H}}_f] & [\bar{\mathbf{H}}_f] & \dots & [\bar{\mathbf{H}}_f] \\ \hline [\bar{\mathbf{H}}_p]_1 & [\bar{\mathbf{H}}_p]_2 & & [\bar{\mathbf{H}}_p]_m \end{array} \right] \begin{Bmatrix} \mathbf{g}_1(x) \\ \mathbf{g}_2(x) \\ \vdots \\ \mathbf{g}_m(x) \end{Bmatrix} \quad (8)$$

where  $\Delta_f(x)$  and  $\Delta_p$  are the vectors of unknown and prescribed nodal displacements, respectively.  $\mathbf{F}_p(x)$  is the vector of unknown forces (support reactions). From eq. (8), the unknown displacements are given by

$$\Delta_f(x) = -[\mathbf{K}_{ff}]^{-1} \left[ [\bar{\mathbf{H}}_f]_1 \quad [\bar{\mathbf{H}}_f]_2 \quad \dots \quad [\bar{\mathbf{H}}_f]_m \right] \begin{Bmatrix} \mathbf{g}_1(x) \\ \mathbf{g}_2(x) \\ \vdots \\ \mathbf{g}_m(x) \end{Bmatrix} - [\mathbf{K}_{ff}]^{-1} [\mathbf{K}_{fp}] \Delta_p \quad (9a)$$

or

$$\Delta_f(x) = [\Delta]_f \begin{Bmatrix} \mathbf{g}_1(x) \\ \mathbf{g}_2(x) \\ \vdots \\ \mathbf{g}_m(x) \end{Bmatrix} - \bar{\Delta} \quad (9b)$$

where

$$[\Delta]_f \equiv [\Delta_f]_1 \quad [\Delta_f]_2 \quad \dots \quad [\Delta_f]_m = -[\mathbf{K}_{ff}]^{-1} \left[ [\bar{\mathbf{H}}_f]_1 \quad [\bar{\mathbf{H}}_f]_2 \quad \dots \quad [\bar{\mathbf{H}}_f]_m \right] \quad (10a)$$

and

$$\bar{\Delta} = -[\mathbf{K}_{ff}]^{-1} [\mathbf{K}_{fp}] \Delta_p \quad (10b)$$

Similarly, the support reactions can be written as

$$\mathbf{F}_p(x) = [\mathbf{F}] \begin{Bmatrix} \mathbf{g}_1(x) \\ \mathbf{g}_2(x) \\ \vdots \\ \mathbf{g}_m(x) \end{Bmatrix} + \bar{\mathbf{F}} \quad (11)$$

where

$$[\mathbf{F}] = [\mathbf{K}_{pf}] [\Delta]_f + \left[ [\bar{\mathbf{H}}_p]_1 \quad [\bar{\mathbf{H}}_p]_2 \quad \cdots \quad [\bar{\mathbf{H}}_p]_m \right] \quad (12a)$$

and

$$\bar{\mathbf{F}} = [\mathbf{K}_{pp}] \Delta_p - [\mathbf{K}_{pf}] \bar{\Delta} \quad (12b)$$

It should be noticed that  $\bar{\Delta}$  and  $\bar{\mathbf{F}}$  would vanish if the prescribed displacements  $\Delta_p$  are zeros.

Equations (10) and (12) give solutions for nodal displacements and support reactions, which are independent of load position within loaded spans. Expressions for the nodal displacements and support reactions as functions of  $x$ , the load position, can be obtained from eqs. (9) and (11), respectively. For example, as the unit load traverses member  $j$ , the structure nodal displacements are given by

$$\Delta_f(x) = -[\mathbf{K}_{ff}]^{-1} [\bar{\mathbf{H}}_f]_j \mathbf{g}_j(x) - [\mathbf{K}_{ff}]^{-1} [\mathbf{K}_{fp}] \Delta_p \quad (13)$$

where  $x$  changes from 0 to 1 as the load moves from I-node to J-node of member  $j$ . The change in the values of  $x$  can be selected at any rate to reflect the desired load movement increment. Support reactions can be expressed in similar manner using eq. (11).

### Local response

The member nodal displacements in local coordinates,  $\delta(x)$ , can be obtained from the structure nodal displacements (eq. 9) by transformation. Equation (4) can then be used to give the nodal forces of member  $j$  as functions of the load position  $x$

$$\mathbf{f}_j(x) \equiv \begin{Bmatrix} N_I(x) \\ V_I(x) \\ M_I(x) \\ N_J(x) \\ V_J(x) \\ M_J(x) \end{Bmatrix}_j = [\mathbf{k}]_j \boldsymbol{\delta}(x) + \mathbf{f}_j^F(x) \quad (14)$$

where the fixed-end force terms  $\mathbf{f}_j^F(x)$  vanish when the load is not on member  $j$ . The member axial force is simply given by

$$N_j(x) = N_J(x) - N_I(x) \quad (15a)$$

and the shear and moment at a distance  $sl_j$  from the member I-node ( $0 \leq s \leq 1$ ) are given by (see Fig. 2)

$$\begin{aligned} V(s,x) &= V_I(x) - H(s-x) \\ M(s,x) &= -M_I(x) + s(V_I(x) - H(s-x)) \end{aligned} \quad \text{when the unit load is on the member}$$

and

$$\begin{aligned} V(s,x) &= V_I(x) \\ M(s,x) &= -M_I(x) + sV_I(x) \end{aligned} \quad \text{otherwise} \quad (15b)$$

where  $H(s-x)$  is the step function:

$$H(s-x) = \begin{cases} 1 & \text{when } s > x \\ 0 & \text{otherwise} \end{cases}$$

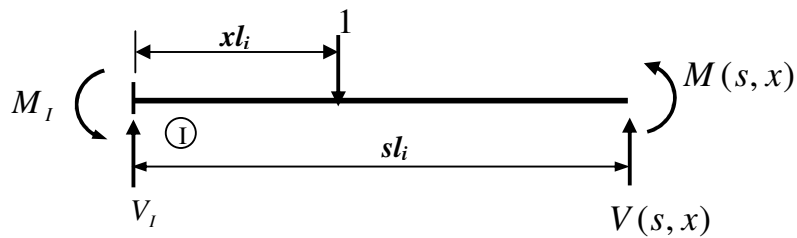


Figure 2: Shear and moment at distance  $sl$  from  $I$  node in member  $j$

The above procedure was effortlessly implemented into a computer program using MATLAB. The steps involved for determining the influence lines are summarized as follows:

- step 1: Construct the structure stiffness matrix  $[\mathbf{K}]$
- step 2: Compute the matrix  $[\mathbf{H}]_j$  (eq. 6) and assemble into the load matrix  $[\bar{\mathbf{H}}]_j, j = 1, \dots, m$
- step 3: Partition the system equation as shown in eq. (8) and solve for  $[\Delta]$  and  $[\mathbf{F}]$  (eqs. 10 and 12)\* .
- step 4: Obtain reactions influence coefficients (ordinates of influence lines), if required, by substituting values for  $x$  in eq. (11).
- step 5: Compute the influence coefficients of a member axial force, shear or moment at any location within a member by substituting values for  $s$  and  $x$  into eqs. (14-15).

Note that  $[\Delta]$  and  $[\mathbf{F}]$  are the results of a standard static stiffness matrix analysis. The desired influence lines are subsequently computed from these results using member stiffness equations and simple statics. It may be worth mentioning that in the computer program, the abscissas of the influence lines can be arranged to reflect the actual positions of the load with respect to a reference point. The reference point could be the point at which the load would start traversing the structure.

## COMPARISON WITH OTHER METHODS

The determination of influence lines can be based on positioning unit loads at several locations on the structure, the adjoint method [2] or the Muller-Breslau principle [3]. All of the aforementioned methods as well as the present method are implemented using the standard finite element formulation, namely the displacement method. The effectiveness of a method against the others is a function of the CPU time, preparation of data and the flexibility of the method. The CPU time that a method needs to obtain a set of influence lines for a structure depends mainly on the required number of rounds of the FE analysis and the degrees of freedom of the structure. Obviously the degrees of freedom of the structure would increase if the method requires member subdivisions. The effort needed for preparation of data and the robust and flexibility of the method are also of great importance to the user. Therefore, the aforementioned criteria can be used to assess the effectiveness of the present method.

Table 1 shows a comparison of the present method against the Adjoint and Muller-Breslau methods. It is evident from the Table that the present method is superior in all comparison criteria which proves the efficiency of the method.

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\* Note that inversion of the stiffness matrix as implied by eq. (10) is not required. Instead, direct reduction of the system of equations using standard elimination procedure is more efficient.

## NUMERICAL EXAMPLES

Two example problems are considered to demonstrate the efficiency of the present procedure. The first is a four-span continuous beam and the second is a rigid frame that resembles a typical bridge structure. The problems are solved using a computer program that was developed based on the procedure described in this paper.

### *Beam Problem*

The continuous beam shown in Fig. 3a has been considered by some authors [3,4] to demonstrate different methods for determining influence lines. Belegundu [3] only presented the influence line for the moment at support B using the adjoint method. With the present method, the influence lines for any action (reaction, shear, moment, axial force) at any point in the structure for any load movement can be obtained in one cycle stiffness matrix analysis in conjunction with simple statics. The input data for this problem are those required for a standard stiffness matrix analysis plus the specification of the members that would be traversed by the moving load (members 1,2,3 and 4 in this problem).

The output of the program includes tabulated values of influence ordinates and plots of influence lines. Due to space limitation, only plots of influence lines are presented. The influence lines for support reactions, shears and moments at selected points in the beam are shown in Figs. 4b-d. Obviously other influence lines, if needed, can easily be obtained without repeating the stiffness matrix analysis.

### *Frame Problem*

The rigid frame shown in Fig. 4a is typical in bridge structures [7]. The dimensions shown in the figure and member properties are chosen to reflect those of real bridges of this type. Once again, the input data for this problem are the data required for a standard stiffness matrix analysis plus the specification of the members that would be traversed by the moving load (members 1,2 and 3 in this problem).

The influence lines for moments and shears in spans AB and BC are presented in Figs. 4b-e. The curves shown describes moments and shears at ( $3m, 6m, \dots, l$ ) in the spans. Figure 4f shows the influence lines for moment, shear and axial force in member BE.

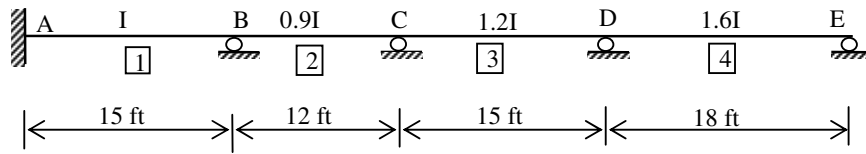
## CONCLUSION

A new procedure for computing the influence lines for beams and plane frames is presented. The procedure is based on the standard stiffness matrix analysis of

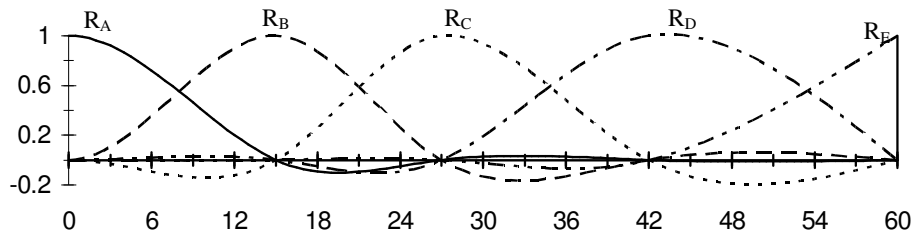
the structure in conjunction with simple statics. The efficiency of the procedure comes from the fact that it does not require the subdivision of members nor the specification of load movement increment prior to the stiffness matrix analysis. It needs only the specification of members that would be traversed by the moving load. After the stiffness matrix analysis is performed, the computation of influence lines for any action at any point in the structure with any load movement increment is accomplished by simple calculations. This gives the user the ability to study and precisely locate the critical sections in the structure when subjected to moving loads. The procedure was applied to some building and bridges structures (two of them are shown in this paper) and proved to be precise and robust. The extension of the procedure to compute forces and moments envelopes is underway and will be presented in upcoming paper.

Table 1: Comparison of the present method with the Adjoint and Muller-Breslau methods.

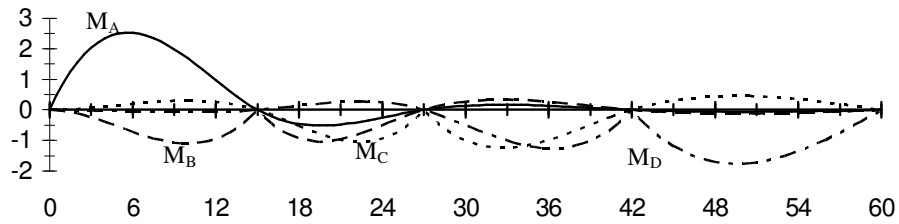
		Adjoint method [2]	Muller-Breslau [3]	Present Method
CPU time	Member subdivision	Needed for I.Ls. within a member	Needed for all cases	Not needed.
	No. of rounds of FE analysis	One for each I.L. within a member	One for each I.L.	One for all I.Ls.
Data preparation		Needs new data for each I.L. within a member	Needs new data for each I.L.	Once for all I.Ls. (original structure data)
Robust and flexibility		Desired I.L. needs to be determined prior to FE analysis (input data depends on the required I.Ls.)	Same as adjoint method.	All desired I.Ls. can be obtained after FE analysis using simple statics.



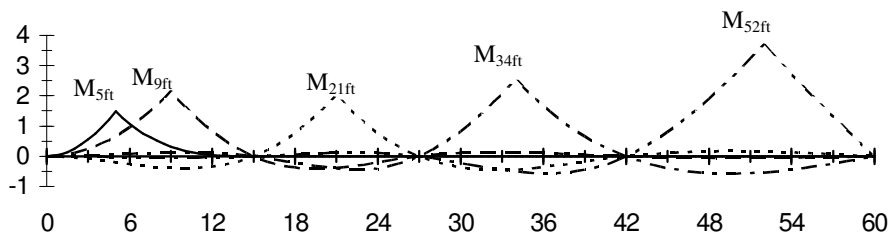
(a) Continuous beam



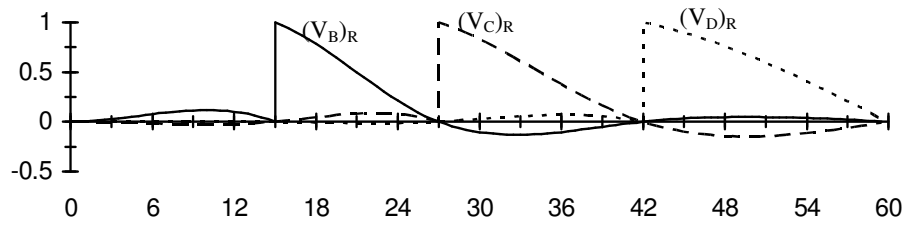
(b) Influence lines for support reactions



(c) Influence lines for moments at supports

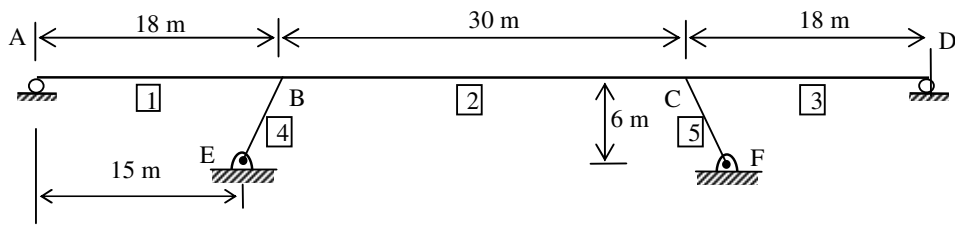


(d) Influence lines for moments at specified locations within beam spans.

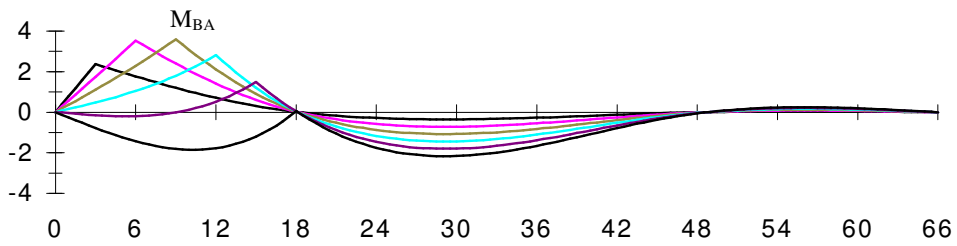


(e) Influence lines for shears just to the right of supports B, C and D.

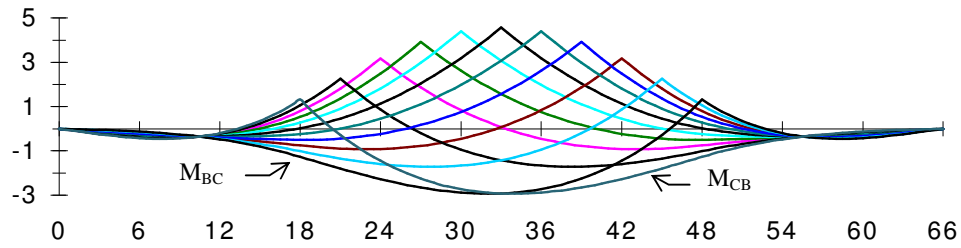
Figure 3: Influence lines for a beam problem.



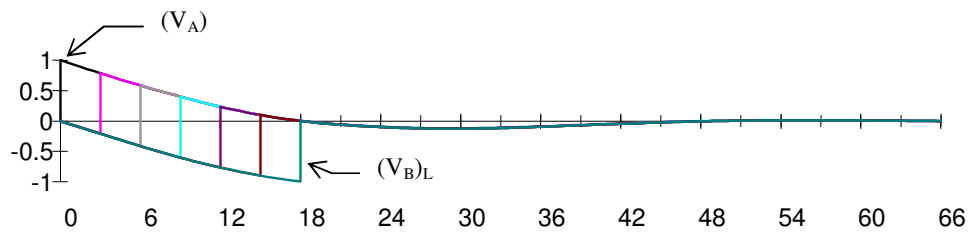
(a) Rigid frame structure



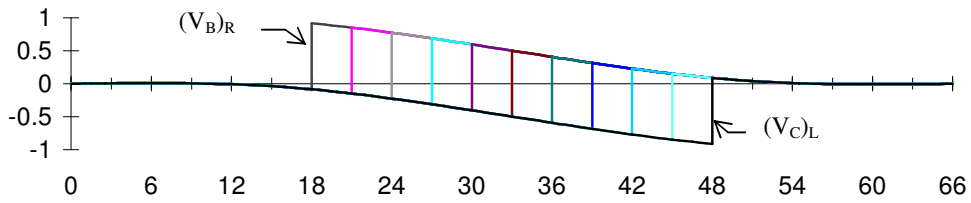
(b) Influence lines for moments in span AB.



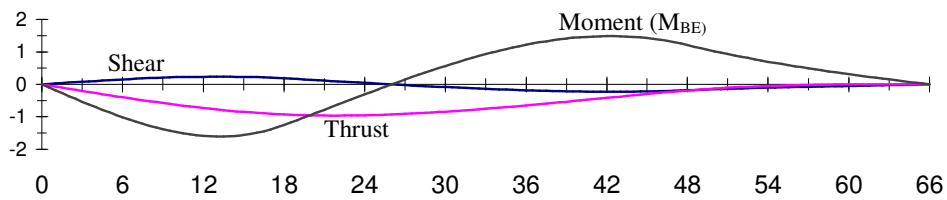
(c) Influence lines for moments in spans BC.



(d) Influence lines for shear in spans AB.



(e) Influence lines for shear in spans BC.



(f) Influence lines for moment, shear and thrust in column BE

Figure 4: Influence Lines for a Frame Problem. *Note: The influence lines shown for spans AB and BC represent the moments and shears at 3m-intervals.*

## REFERENCES

- [1] Hsieh, Y., "*Elementary Theory of Structures*", Prentice-Hall, Inc., Englewood Cliffs, N. J., 1970.
- [2] Hibbeler, R. C., "*Structural Analysis*", Prentice-Hall, Inc., Englewood Cliffs, N. J., 1995.
- [3] Belegundu, A. D., "The Adjoint Method for Determining Influence Lines", *Computers & Structures*, Vol. 29, No. 2, 345-350, 1988.
- [4] Cifuentes, A. and Paz, M., "A Note on the Determination of Influence Lines and Surfaces Using Finite Elements", *Finite Elements in Analysis and Design*, Vol. 7, 299-305, 1991.
- [5] Washington State Department of Transportation, *QconBridge Software*, Version 1.0c-March, 1997.
- [6] McGuire, W. and Gallagher, R. H., "*Matrix Structural Analysis*", John Wiley & Sons, Inc., New York, 1979.
- [7] Heins, C. P. and Lawrie, R. A., "*Design of Modern Highway Bridges*", John Wiley & Sons, Inc., New York, 1984.